

The effective Hamiltonian for thin layers with non-Hermitian Robin-type boundary conditions

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24 February 2011

Abstract

The Laplacian in an unbounded tubular neighbourhood of a hyperplane with non-Hermitian complex-symmetric Robin-type boundary conditions is investigated in the limit when the width of the neighbourhood diminishes. We show that the Laplacian converges in a norm resolvent sense to a self-adjoint Schrödinger operator in the hyperplane whose potential is expressed solely in terms of the boundary coupling function. As a consequence, we are able to explain some peculiar spectral properties of the non-Hermitian Laplacian by known results for Schrödinger operators.

1 Introduction

There has been a growing interest in spectral properties of differential operators in shrinking tubular neighbourhoods of submanifolds of Riemannian manifolds, subject to various boundary conditions. This is partly motivated by the enormous progress in semiconductor physics, where it is reasonable to try to model a complicated quantum Hamiltonian in a thin nanostructure by an effective operator in a lower dimensional substrate. But the problem is interesting from the purely mathematical point of view as well, because one deals with a singular limit and it is not always obvious how the information about the geometry and boundary conditions are transformed into coefficients of the effective Hamiltonian.

The interest has mainly focused on self-adjoint problems, namely on the Laplacian in the tubular neighbourhoods with uniform boundary conditions of Dirichlet [6, 5, 8, 17] or Neumann [19] or a combination of those [15]. For more references see the review article [11]. The purpose of the present paper is to show that one may obtain an interesting self-adjoint effective operator in the singular limit even if the initial operator is not Hermitian and the geometry is rather trivial.

We consider an operator H_ε which acts as the Laplacian in a d -dimensional layer:

$$H_\varepsilon u = -\Delta u \quad \text{in} \quad \Omega_\varepsilon := \mathbb{R}^{d-1} \times (0, \varepsilon), \quad (1.1)$$

where $d \geq 2$ and ε is a small positive parameter, subjected to non-Hermitian boundary conditions on $\partial\Omega_\varepsilon$. Instead of considering the general problem, we rather restrict to a special case of separated Robin-type boundary conditions

$$\frac{\partial u}{\partial x_d} + i\alpha(x')u = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \quad (1.2)$$

where $x = (x_1, \dots, x_{d-1}, x_d) \equiv (x', x_d)$ denotes a generic point in Ω_ε and α is a real-valued bounded function. More precisely, we consider H_ε as the m-sectorial operator H_ε on $L_2(\Omega_\varepsilon)$ which acts as (1.1) in the distributional sense on the domain consisting of functions u from the Sobolev space $W_2^2(\Omega_\varepsilon)$ satisfying the boundary conditions (1.2). We postpone the formal definition to the following section.

The model H_ε in $d = 2$ was introduced in [4] by the present authors. In that paper, we developed a perturbation theory to study spectral properties of H_ε with ε fixed in the regime when α represents a small and local perturbation of constant α_0 (see below for the discussion of some of the results). Additional spectral properties of H_ε were further studied in [16] by numerical methods. The present paper can be viewed as an addendum by keeping α (and d) arbitrary but sending rather the layer width ε to zero. We believe that the present convergence results provide a valuable insight into the spectral phenomena observed in the two previous papers.

The particular feature of the choice (1.2) is that the boundary conditions are \mathcal{PT} -symmetric in the sense that H_ε commutes with the product operator \mathcal{PT} . Here \mathcal{P} denotes the parity (space) reversal operator $(\mathcal{P}u)(x) := u(x', \varepsilon - x_d)$ and \mathcal{T} stands for the complex conjugation $(\mathcal{T}u)(x) := \overline{u(x)}$; the latter can be understood as the time reversal operator in the framework of quantum mechanics. The relevance of non-Hermitian \mathcal{PT} -symmetric models in physics has been discussed in many papers recently, see the review articles [1, 18]. Non-Hermitian boundary conditions of the type (1.2) were considered in [13] to model open (dissipative) quantum systems. The role of (1.2) with constant α in the context of perfect-transmission scattering in quantum mechanics is discussed in [12].

Another feature of (1.2) is that the spectrum of H_ε “does not explode” as the layer shrinks, meaning precisely that the resolvent operator $(H_\varepsilon + 1)^{-1}$ admits a non-trivial limit in $L_2(\Omega_\varepsilon)$ as $\varepsilon \rightarrow 0$. As a matter of fact, it is the objective of the present paper to show that H_ε converges in a norm resolvent sense to the $(d - 1)$ -dimensional operator

$$H_0 := -\Delta + \alpha^2 \quad \text{on} \quad L_2(\mathbb{R}^{d-1}), \quad (1.3)$$

which is a self-adjoint operator on the domain $W_2^2(\mathbb{R}^{d-1})$. Again, we postpone the precise statement of the convergence, which has to take into account that the operators H_ε and H_0 act on different Hilbert spaces, till the following section (*cf* Theorem 2.1). However, let us comment on spectral consequences of the result already now.

First of all, we observe that a significantly non-self-adjoint operator H_ε converges, in the norm resolvent sense, to a self-adjoint Schrödinger operator H_0 . The latter contains the information about the non-self-adjoint boundary conditions of the former in a simple potential term. It follows from general facts [14, Sec. IV.3.5] that discrete eigenvalues of H_ε either converge to discrete eigenvalues of H_0 or go to complex infinity or to the essential spectrum of H_0 as $\varepsilon \rightarrow \infty$.

In particular, assuming that H_ε and H_0 have the same essential spectrum (independent as a set of ε), the spectrum of H_ε must approach the real axis (or go to complex infinity) in the limit as $\varepsilon \rightarrow 0$. Although numerical computations performed in [16] suggest that H_ε might have complex spectra in general, perturbation analysis developed in [4] for the 2-dimensional case shows that both the essential spectrum and weakly coupled eigenvalues are real. The present paper demonstrates that the spectrum is real also as the layer becomes infinitesimally thin, for every $d \geq 2$. We would like to stress that the \mathcal{PT} -symmetry itself is not sufficient to ensure the reality of the spectrum and that the proof that a non-self-adjoint operator has a real spectrum is a difficult task.

It is also worth noticing that the limiting operator H_0 provides quite precise information about the spectrum of H_ε in the weak coupling regime for $d = 2$ and $|\alpha| < \pi/\varepsilon$ (ε fixed). Indeed, let us consider the following special profile of the boundary function:

$$\alpha_c(x') := \alpha_0 + c\beta(x'),$$

where α_0 is a real constant, β is a real-valued function of compact support and c is a real parameter (the regime of weak coupling corresponds to small c). Note that the essential spectrum of both H_ε and H_0 coincides with the interval $[\alpha_0^2, \infty)$, for β is compactly

supported (*cf* [4, Thm. 2.2]). It is proved in [4] that if $\alpha_0 \int_{\mathbb{R}} \beta(x') dx'$ is negative, then H_ε possesses exactly one discrete real eigenvalue $\mu(c)$ converging to α_0^2 as $c \rightarrow 0+$ and the asymptotic expansion

$$\mu(c) = \alpha_0^2 - c^2 \alpha_0^2 \left(\int_{\mathbb{R}} \beta(x') dx' \right)^2 + \mathcal{O}(c^3) \quad \text{as} \quad c \rightarrow 0+$$

holds true. As a converse result, it is proved in [4] that there is no such a weakly coupled eigenvalue if the quantity $\alpha_0 \int_{\mathbb{R}} \beta(x') dx'$ is positive. These weak coupling properties, including the asymptotics above, are well known for the Schrödinger operator H_0 with the potential given by α_c^2 , see [9].

At the same time, the form of the potential in H_0 explains some of the peculiar characteristics of H_ε even for large c . As an example, let us recall that a highly non-monotone dependence of the eigenvalues of H_ε on the coupling parameter c was observed in the numerical analysis of [16]. As the parameter increases, a real eigenvalue typically emerges from the essential spectrum, reaches a minimum and then comes back to the essential spectrum again. This behaviour is now easy to understand from the non-linear dependence of the potential α_c^2 on c .

On the other hand, we cannot expect that H_0 represents a good approximation of H_ε for the values of parameters for which H_ε is known to possess complex eigenvalues [16]. It would be then desirable to compute the next to leading term in the asymptotic expansion of H_ε as $\varepsilon \rightarrow 0$.

This paper is organized as follows. In Section 2 we give a precise definition of the operators H_ε and H_0 and state the norm resolvent convergence of the former to the latter as $\varepsilon \rightarrow 0$ (Theorem 2.1). The rest of the paper consists of Section 3 in which a proof of the convergence result is given.

2 The main result

We start with giving a precise definition of the operators H_ε and H_0 .

The limiting operator (1.3) can be immediately introduced as a bounded perturbation of the free Hamiltonian on $L_2(\mathbb{R}^{d-1})$, which is well known to be self-adjoint on the domain $W_2^2(\mathbb{R}^{d-1})$. For later purposes, however, we equivalently understand H_0 as the operator associated on $L_2(\mathbb{R}^{d-1})$ with the quadratic form

$$h_0[v] := \int_{\mathbb{R}^{d-1}} |\nabla' v(x')|^2 dx' + \int_{\mathbb{R}^{d-1}} \alpha(x')^2 |v(x')| dx', \\ v \in \mathfrak{D}(h_0) := W_2^1(\mathbb{R}^{d-1}).$$

Here and in the sequel we denote by ∇' the gradient operator in \mathbb{R}^{d-1} , while ∇ stands for the “full” gradient in \mathbb{R}^d .

In the same manner, we introduce H_ε as the m-sectorial operator associated on $L_2(\Omega_\varepsilon)$ with the quadratic form

$$h_\varepsilon[u] := \int_{\Omega_\varepsilon} |\nabla u(x)|^2 dx + i \int_{\mathbb{R}^{d-1}} \alpha(x') |u(x', \varepsilon)|^2 dx' - i \int_{\mathbb{R}^{d-1}} \alpha(x') |u(x', 0)|^2 dx', \\ u \in \mathfrak{D}(h_0) := W_2^1(\Omega_\varepsilon).$$

Here the boundary terms are understood in the sense of traces. Note that H_ε is not self-adjoint unless $\alpha = 0$ (in this case H_ε coincides with the Neumann Laplacian in the layer Ω_ε). The adjoint of H_ε is determined by simply changing α to $-\alpha$ (or i to $-i$) in the definition of h_ε . Moreover, H_ε is \mathcal{T} -self-adjoint [7, Sec. III.5] (or complex-symmetric [10]), *i.e.* $H_\varepsilon^* = \mathcal{T} H_\varepsilon \mathcal{T}$.

The form h_ε is well defined under the mere condition that α is bounded. However, if we strengthen the regularity to $\alpha \in W_\infty^1(\mathbb{R}^{d-1})$, it can be shown by standard procedures (*cf* [4, Sec. 3]) that H_ε coincides with the operator described in the introduction, *i.e.*,

it acts as the (distributional) Laplacian (1.1) on the domain formed by the functions u from $W_2^2(\Omega_\varepsilon)$ satisfying the boundary conditions (1.2) in the sense of traces.

The operator H_0 is clearly non-negative. An analogous property for H_ε is contained in the following result

$$\sigma(H_\varepsilon) \subset \left\{ z \in \mathbb{C} \mid \operatorname{Re} z \geq 0, |\operatorname{Im} z| \leq 2 \|\alpha\|_\infty \sqrt{\operatorname{Re} z} \right\}. \quad (2.1)$$

Here and in the sequel we denote by $\|\cdot\|_\infty$ the supremum norm. (2.1) can be proved exactly in the same way as in [4, Lem. 3.1] for $d = 2$ by estimating the numerical range of H_ε . In particular, the open left half-plane of \mathbb{C} belongs to the resolvent set of both H_ε and H_0 .

Another general spectral property of H_ε , common with H_0 , is that its residual spectrum is empty. This is a consequence of the \mathcal{T} -self-adjointness property of H_ε as pointed out in [4, Corol. 2.1].

Since H_ε and H_0 act on different Hilbert spaces, we need to explain how the convergence of the corresponding resolvent operators is understood. We decompose our Hilbert space into an orthogonal sum

$$L_2(\Omega_\varepsilon) = \mathfrak{H}_\varepsilon \oplus \mathfrak{H}_\varepsilon^\perp, \quad (2.2)$$

where the subspace \mathfrak{H}_ε consists of functions from $L_2(\Omega_\varepsilon)$ of the form $x \mapsto \psi(x')$, *i.e.* independent of the “transverse” variable x_d . The corresponding projection is given by

$$(P_\varepsilon u)(x) := \frac{1}{\varepsilon} \int_0^\varepsilon u(x) dx_d \quad (2.3)$$

and it can be viewed as a projection onto a constant function in the transverse variable. We also write $P_\varepsilon^\perp := I - P_\varepsilon$. Since the functions from \mathfrak{H}_ε depend on the “longitudinal” variables x' only, \mathfrak{H}_ε can be naturally identified with $L_2(\mathbb{R}^{d-1})$. Hence, with an abuse of notations, we may identify any operator on $L_2(\mathbb{R}^{d-1})$ as the operator acting on $\mathfrak{H}_\varepsilon \subset L_2(\Omega_\varepsilon)$, and vice versa.

The norm and the inner product in $L_2(\Omega_\varepsilon)$ will be denoted by $\|\cdot\|_\varepsilon$ and $(\cdot, \cdot)_\varepsilon$, respectively. We keep the same notation $\|\cdot\|_\varepsilon$ for the operator norm on $L_2(\Omega_\varepsilon)$. The norm and the inner product in $L_2(\mathbb{R}^{d-1})$ will be denoted by $\|\cdot\|$ and (\cdot, \cdot) , respectively, *i.e.* without the subscript ε . All the inner products are assumed to be linear in the first component. Finally, we denote the norm in $W_2^1(\Omega_\varepsilon)$ by $\|\cdot\|_{\varepsilon,1}$ and we keep the same notation for the norm of bounded operators from $L_2(\Omega_\varepsilon)$ to $W_2^1(\Omega_\varepsilon)$.

Now we are in a position to formulate the main result of this paper.

Theorem 2.1. *Assume $\alpha \in W_\infty^1(\mathbb{R}^{d-1})$. Then the inequalities*

$$\|(H_\varepsilon + 1)^{-1} - (H_0 + 1)^{-1} P_\varepsilon\|_\varepsilon \leq C \varepsilon, \quad (2.4)$$

$$\|(H_\varepsilon + 1)^{-1} - (1 + Q)(H_0 + 1)^{-1} P_\varepsilon\|_{\varepsilon,1} \leq C(\varepsilon) \varepsilon \quad (2.5)$$

hold true, where $Q(x) := -i\alpha(x')x_d$ and

$$\begin{aligned} C &:= \sqrt{\frac{1}{\pi^2} + \frac{(\|\nabla' \alpha\|_\infty + 2\|\alpha\|_\infty)^2}{3}}, \\ C(\varepsilon) &:= \sqrt{\frac{1}{\pi^2} + \left(\frac{\|\nabla' \alpha\|_\infty + \|\alpha\|_\infty}{\sqrt{3}} + C_1(\varepsilon) \right)^2}, \\ C_1(\varepsilon) &:= \sqrt{\left(\frac{\varepsilon \|\alpha\|_\infty^2}{2\sqrt{5}} \right)^2 + \left(\frac{\varepsilon \|\alpha\|_\infty^2}{2\sqrt{5}} + \frac{\|\alpha\|_\infty \sqrt{\|\alpha\|_\infty^2 + \|\nabla' \alpha\|_\infty^2 \varepsilon^2}}{\sqrt{3}} \right)^2}. \end{aligned}$$

Let us discuss the result of this theorem. It says that the operator H_ε converges to H_0 in the norm resolvent sense. Note that, contrary to what happens for instance in the case of uniform Dirichlet boundary conditions, here we can choose the spectral parameter fixed (e.g. $-1 \in \rho(H_\varepsilon) \cap \rho(H_0)$ as in the theorem) and still get a non-trivial result.

If we treat the convergence of the resolvents in the topology of bounded operators in $L_2(\Omega_\varepsilon)$, the estimate (2.4) says that the rate of the convergence is of order $\mathcal{O}(\varepsilon)$. At the same time, if we consider the convergence as for the operators acting from $L_2(\Omega_\varepsilon)$ into $W_2^1(\Omega_\varepsilon)$, to keep the same rate of the convergence, one has to use the function Q . This functions is to be understood as a corrector needed to have the convergence in a stronger norm. Such situation is well-known and it often happens for singularly perturbed problems, especially in the homogenization theory, see, e.g., [3, 2, 20].

3 Proof of Theorem 2.1

Throughout this section we assume $\alpha \in W_\infty^1(\mathbb{R}^{d-1})$. With an abuse of notation, we denote by the same symbol α both the function on \mathbb{R}^{d-1} and its natural extension $x \mapsto \alpha(x')$ to \mathbb{R}^d .

We start with two auxiliary lemmata. The first tells us that the subspace $\mathfrak{H}_\varepsilon^\perp$ is negligible for H_ε in the limit as $\varepsilon \rightarrow 0$.

Lemma 3.1. *For any $f \in L_2(\Omega_\varepsilon)$, we have*

$$\|(H_\varepsilon + 1)^{-1} P_\varepsilon^\perp f\|_{\varepsilon,1} \leq \frac{\varepsilon}{\pi} \|P_\varepsilon^\perp f\|_\varepsilon. \quad (3.1)$$

Proof. For any fixed $f \in L_2(\Omega_\varepsilon)$, let us set $u := (H_\varepsilon + 1)^{-1} P_\varepsilon^\perp f \in \mathfrak{D}(H_\varepsilon) \subset W_2^1(\Omega_\varepsilon)$. In other words, u satisfies the resolvent equation

$$\forall v \in W_2^1(\Omega_\varepsilon), \quad h_\varepsilon(u, v) + (u, v)_\varepsilon = (P_\varepsilon^\perp f, v)_\varepsilon,$$

where $h_\varepsilon(\cdot, \cdot)$ denotes the sesquilinear form associated with the quadratic form $h_\varepsilon[\cdot]$. Choosing u for the test function v and taking the real part of the obtained identity, we get

$$\|u\|_{\varepsilon,1}^2 = \operatorname{Re} (P_\varepsilon^\perp f, u)_\varepsilon = \operatorname{Re} (P_\varepsilon^\perp f, P_\varepsilon^\perp u)_\varepsilon \leq \|P_\varepsilon^\perp f\|_\varepsilon \|P_\varepsilon^\perp u\|_\varepsilon. \quad (3.2)$$

Employing the decomposition $u = P_\varepsilon u + P_\varepsilon^\perp u$, the left hand side of (3.2) can be estimated as follows

$$\|u\|_{\varepsilon,1}^2 \geq \|\nabla u\|_\varepsilon^2 \geq \|\partial_d u\|_\varepsilon^2 = \|\partial_d P_\varepsilon^\perp u\|_\varepsilon^2 \geq (\pi/\varepsilon)^2 \|P_\varepsilon^\perp u\|_\varepsilon^2. \quad (3.3)$$

Here the last inequality follows from the variational characterization of the second eigenvalue of the Neumann Laplacian on $L_2((0, \varepsilon))$ and Fubini's theorem. Combining (3.3) with (3.2), we obtain

$$\|P_\varepsilon^\perp u\|_\varepsilon \leq (\varepsilon/\pi)^2 \|P_\varepsilon^\perp f\|_\varepsilon.$$

Finally, applying the obtained inequality to the right hand side of (3.2), we conclude with

$$\|u\|_{\varepsilon,1}^2 \leq (\varepsilon/\pi)^2 \|P_\varepsilon^\perp f\|_\varepsilon^2.$$

This is equivalent to the estimate (3.1). \square

In the second lemma we collect some elementary estimates we shall need later on.

Lemma 3.2. *We have*

$$|e^{-i\alpha x_d} - 1| \leq \|\alpha\|_\infty x_d, \quad (3.4)$$

$$|e^{-i\alpha x_d} - 1 + i\alpha x_d| \leq \frac{1}{2} \|\alpha\|_\infty^2 x_d^2, \quad (3.5)$$

$$|\nabla(e^{-i\alpha x_d} - 1 + i\alpha x_d)| \leq \|\alpha\|_\infty x_d \sqrt{\|\alpha\|_\infty^2 + \|\nabla' \alpha\|_\infty^2 x_d^2}. \quad (3.6)$$

Proof. The estimates (3.4) and (3.5) are elementary and we leave the proofs to the reader. The last estimate (3.6) follows from (3.4) and the identity

$$\nabla(e^{-i\alpha x_d} - 1 + i\alpha x_d) = i(1 - e^{-i\alpha x_d}) \begin{pmatrix} x_d \nabla' \alpha \\ \alpha \end{pmatrix}$$

taken into account. \square

We continue with the proof of Theorem 2.1. Let $f \in L_2(\Omega_\varepsilon)$. Accordingly to (2.2), f admits the decomposition $f = P_\varepsilon f + P_\varepsilon^\perp f$ and we have

$$\|f\|_\varepsilon^2 = \|P_\varepsilon f\|_\varepsilon^2 + \|P_\varepsilon^\perp f\|_\varepsilon^2 = \varepsilon \|P_\varepsilon f\|^2 + \|P_\varepsilon^\perp f\|_\varepsilon^2. \quad (3.7)$$

We define $u := (H_\varepsilon + 1)^{-1}f$ and make the decomposition

$$u = u_0 + u_1 \quad \text{with} \quad u_0 := (H_\varepsilon + 1)^{-1}P_\varepsilon f, \quad u_1 := (H_\varepsilon + 1)^{-1}P_\varepsilon^\perp f. \quad (3.8)$$

In view of Lemma 3.1, u_1 is negligible in the limit as $\varepsilon \rightarrow 0$,

$$\|u_1\|_{\varepsilon,1} \leq \frac{\varepsilon}{\pi} \|P_\varepsilon^\perp f\|_\varepsilon. \quad (3.9)$$

It remains to study the dependence of u_0 on ε . We construct u_0 as follows

$$u_0(x) = e^{-i\alpha(x')x_d} w_0(x') + w_1(x), \quad \text{where} \quad w_0 := (H_0 + 1)^{-1}P_\varepsilon f \quad (3.10)$$

and w_1 is a function defined by this decomposition.

First, we establish a rather elementary bound for w_0 .

Lemma 3.3. *We have*

$$\|w_0\|_{\varepsilon,1} \leq \|P_\varepsilon f\|_\varepsilon.$$

Proof. By definition, w_0 satisfies the resolvent equation

$$\forall v \in W_2^1(\mathbb{R}^{d-1}), \quad h_0(w_0, v) + (w_0, v) = (P_\varepsilon f, v), \quad (3.11)$$

where $h_0(\cdot, \cdot)$ denotes the sesquilinear form associated with the quadratic form $h_0[\cdot]$. Choosing w_0 for the test function v , we get

$$\|\nabla' w_0\|^2 + \|\alpha w_0\|^2 + \|w_0\|^2 = (P_\varepsilon f, w_0) \leq \|P_\varepsilon f\| \|w_0\|. \quad (3.12)$$

In particular,

$$\|w_0\| \leq \|P_\varepsilon f\|.$$

Using this estimate in the right hand side of (3.12), we get

$$\|\nabla' w_0\|^2 + \|w_0\|^2 \leq \|P_\varepsilon f\|^2.$$

Reintegrating this inequality over $(0, \varepsilon)$, we conclude with the desired bound in Ω_ε . \square

It is more difficult to get a bound for w_1 .

Lemma 3.4. *We have*

$$\|w_1\|_{\varepsilon,1} \leq C_1 \varepsilon \|P_\varepsilon f\|_\varepsilon \quad \text{with} \quad C_0 := \frac{\|\nabla' \alpha\|_\infty + \|\alpha\|_\infty}{\sqrt{3}}.$$

Proof. By definition, u_0 satisfies the resolvent equation

$$\forall v \in W_2^1(\Omega_\varepsilon), \quad h_\varepsilon(u_0, v) + (u_0, v)_\varepsilon = (P_\varepsilon f, v)_\varepsilon.$$

Choosing w_1 for the test function v and using the decomposition (3.10), we get

$$h_\varepsilon[w_1] + \|w_1\|_\varepsilon^2 = (P_\varepsilon f, w_1)_\varepsilon - h_\varepsilon(u_0 - w_1, w_1) - (u_0 - w_1, w_1)_\varepsilon =: F_\varepsilon. \quad (3.13)$$

It is straightforward to check that

$$\begin{aligned} h_\varepsilon(u_0 - w_1, w_1) &= (\nabla w_0, \nabla e^{i\alpha x_d} w_1)_\varepsilon + (\alpha^2 w_0, e^{i\alpha x_d} w_1)_\varepsilon - F'_\varepsilon \\ &= -(w_0, e^{i\alpha x_d} w_1)_\varepsilon + (P_\varepsilon f, e^{i\alpha x_d} w_1)_\varepsilon - F'_\varepsilon \end{aligned}$$

with

$$F'_\varepsilon := i(x_d w_0 \nabla' \alpha, e^{i\alpha x_d} \nabla' w_1)_\varepsilon - i(\nabla' w_0, x_d e^{i\alpha x_d} w_1 \nabla' \alpha)_\varepsilon.$$

Here the first equality follows by algebraic manipulations using an integration by parts, while the second is a consequence of (3.11), with $x' \mapsto e^{i\alpha(x')x_d} w_1(x', x_d)$ being the test function, and Fubini's theorem. At the same time, $(u_0 - w_1, w_1)_\varepsilon = (w_0, e^{i\alpha x_d} w_1)_\varepsilon$. Hence,

$$F_\varepsilon = F'_\varepsilon + (P_\varepsilon f, w_1 - e^{i\alpha x_d} w_1)_\varepsilon.$$

We proceed with estimating F_ε :

$$\begin{aligned} |F_\varepsilon| &\leq \frac{\varepsilon^{3/2}}{\sqrt{3}} \left(\|\nabla' \alpha\|_\infty \|w_0\| \|\nabla' w_1\|_\varepsilon + \|\nabla' \alpha\|_\infty \|\nabla' w_0\| \|w_1\|_\varepsilon + \|\alpha\|_\infty \|P_\varepsilon f\| \|w_1\|_\varepsilon \right) \\ &\leq \frac{\varepsilon^{3/2}}{\sqrt{3}} \left(\|\nabla' \alpha\|_\infty \sqrt{\|w_0\|^2 + \|\nabla' w_0\|^2} \sqrt{\|\nabla' w_1\|_\varepsilon^2 + \|w_1\|_\varepsilon^2} + \|\alpha\|_\infty \|P_\varepsilon f\| \|w_1\|_\varepsilon \right) \\ &\leq \frac{\varepsilon}{\sqrt{3}} \left(\|\nabla' \alpha\|_\infty \|w_0\|_{\varepsilon,1} + \|\alpha\|_\infty \|P_\varepsilon f\|_\varepsilon \right) \|w_1\|_{\varepsilon,1}. \end{aligned}$$

Here the first inequality follows by the Schwarz inequality, an explicit value of the integral of x_d^2 and obvious bounds such as (3.4).

Finally, taking the real part of (3.13) and using the above estimate of $|F_\varepsilon|$, we get

$$\|w_1\|_{\varepsilon,1} \leq \frac{\varepsilon}{\sqrt{3}} \left(\|\nabla' \alpha\|_\infty \|w_0\|_{\varepsilon,1} + \|\alpha\|_\infty \|P_\varepsilon f\|_\varepsilon \right).$$

The desired bound then follows by estimating $\|w_0\|_{\varepsilon,1}$ by means of Lemma 3.3. \square

Now we are in a position to conclude the proof of Theorem 2.1 by simply comparing u with w_0 . As for the convergence in the topology of $L_2(\Omega_\varepsilon)$, we write

$$\begin{aligned} \|u - w_0\|_\varepsilon &= \|u_1 + w_1 + (e^{-i\alpha x_d} - 1)w_0\|_\varepsilon \\ &\leq \|u_1\|_\varepsilon + \|w_1\|_\varepsilon + \|(e^{-i\alpha x_d} - 1)w_0\|_\varepsilon. \end{aligned}$$

Here the last term can be estimated using Lemma 3.2 as follows

$$\|(e^{-i\alpha x_d} - 1)w_0\|_\varepsilon \leq \frac{\varepsilon^{3/2}}{\sqrt{3}} \|\alpha\|_\infty \|w_0\| = \frac{\varepsilon}{\sqrt{3}} \|\alpha\|_\infty \|w_0\|_\varepsilon.$$

Hence, using (3.9), Lemma 3.4 and Lemma 3.3, we get the bound

$$\begin{aligned} \|u - w_0\|_\varepsilon &\leq \|u_1\|_{\varepsilon,1} + \|w_1\|_{\varepsilon,1} + \frac{\varepsilon}{\sqrt{3}} \|\alpha\|_\infty \|w_0\|_\varepsilon \\ &\leq \varepsilon \left(\frac{1}{\pi} \|P_\varepsilon^\perp f\|_\varepsilon + \frac{1}{\sqrt{3}} (\|\nabla' \alpha\|_\infty + 2\|\alpha\|_\infty) \|P_\varepsilon f\|_\varepsilon \right) \\ &\leq C \varepsilon \|f\|_\varepsilon \end{aligned}$$

Here the last estimate follows by the Schwarz inequality recalling (3.7) and holds with the constant C as defined in Theorem 2.1. This proves (2.4).

As for the bound (2.5), we have

$$\begin{aligned} \|u - (1 + Q)w_0\|_{\varepsilon,1} &= \|u_1 + w_1 + (e^{-i\alpha x_d} - 1 + i\alpha x_d)w_0\|_{\varepsilon,1} \\ &\leq \|u_1\|_{\varepsilon,1} + \|w_1\|_{\varepsilon,1} + \|(e^{-i\alpha x_d} - 1 + i\alpha x_d)w_0\|_{\varepsilon,1}. \end{aligned}$$

Here the last term can be estimated using Lemma 3.2 as follows. Employing the individual estimates

$$\begin{aligned} \|(e^{-i\alpha x_d} - 1 + i\alpha x_d)w_0\|_\varepsilon &\leq \frac{\varepsilon^{5/2}}{2\sqrt{5}} \|\alpha\|_\infty^2 \|w_0\| = \frac{\varepsilon^2}{2\sqrt{5}} \|\alpha\|_\infty^2 \|w_0\|_\varepsilon, \\ \|(e^{-i\alpha x_d} - 1 + i\alpha x_d)\nabla w_0\|_\varepsilon &\leq \frac{\varepsilon^{5/2}}{2\sqrt{5}} \|\alpha\|_\infty^2 \|\nabla' w_0\| = \frac{\varepsilon^2}{2\sqrt{5}} \|\alpha\|_\infty^2 \|\nabla' w_0\|_\varepsilon, \\ \|w_0 \nabla(e^{-i\alpha x_d} - 1 + i\alpha x_d)\|_\varepsilon &\leq \frac{\varepsilon^{3/2}}{\sqrt{3}} \|\alpha\|_\infty \sqrt{\|\alpha\|_\infty^2 + \|\nabla' \alpha\|_\infty^2} \varepsilon^2 \|w_0\| \\ &= \frac{\varepsilon}{\sqrt{3}} \|\alpha\|_\infty \sqrt{\|\alpha\|_\infty^2 + \|\nabla' \alpha\|_\infty^2} \varepsilon^2 \|w_0\|_\varepsilon, \end{aligned}$$

and the Schwarz inequality, we may write

$$\|(e^{-i\alpha x_d} - 1 + i\alpha x_d)w_0\|_{\varepsilon,1} \leq C_1(\varepsilon) \varepsilon \|w_0\|_{\varepsilon,1}$$

with the same constant $C_1(\varepsilon)$ as defined in Theorem 2.1. Consequently, using (3.9), Lemma 3.4, Lemma 3.3 and the Schwarz inequality employing (3.7), we get the bound

$$\|u - (1 + Q)w_0\|_{\varepsilon,1} \leq C(\varepsilon) \varepsilon \|f\|_\varepsilon$$

with

$$C(\varepsilon) := \sqrt{\frac{1}{\pi^2} + (C_0 + C_1(\varepsilon))^2}. \quad (3.14)$$

Note that $C(\varepsilon)$ coincides with the corresponding constant of Theorem 2.1. This concludes the proof of Theorem 2.1.

Acknowledgments

D.B. acknowledges the support and hospitality of the Basque Center for Applied Mathematics in Bilbao. D.B. was partially supported by RFBR, by the grants of the President of Russia for young scientists-doctors of sciences (MD-453.2010.1) and for Leading Scientific School (NSh-6249.2010.1), and by the Federal Task Program “Research and educational professional community of innovation Russia” (contract 02.740.11.0612). D.K. was partially supported by the Czech Ministry of Education, Youth and Sports within the project LC06002 and by the GACR grant P203/11/0701.

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